Available online at www.sciencedirect.com

ScienceDirect

Computers and Mathematics with Applications 53 (2007) 437–451

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Stabilization to equilibria of compressible Navier–Stokes equations with infinite mass

A. Novotný[a,*], M. Pokorný[b]

[a] *Université du Sud Toulon-Var, Laboratoire ANAM, B.P. 132, 83957 La Garde-Toulon, France*
[b] *Mathematical Institute of Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic*

Received 13 September 2005; accepted 27 February 2006

Abstract

We consider the compressible Navier–Stokes equations in an exterior three-dimensional domain with non-zero constant density prescribed at infinity. We assume that $p(\varrho) = \varrho^\gamma$, $\gamma > \frac{3}{2}$, and that the force is potential. We show that for time tending to infinity, the density approaches the unique solution to the stationary problem, provided the potential satisfies certain regularity and structural assumptions.

Keywords: Compressible Navier–Stokes equations; Stabilization to equilibria; Stationary compressible Navier–Stokes equations; Compensated compactness

1. Introduction

In this paper, we consider the compressible Navier–Stokes equations in a barotropic regime in an exterior domain in $\mathbb{R}^3$ with a nonzero constant density prescribed at infinity. More precisely, we are concerned with weak solutions to the problem

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} - (\mu + \lambda)\nabla \operatorname{div} \boldsymbol{u} + \nabla p(\varrho) = \varrho \boldsymbol{f} \\ &\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{u}) = 0 \end{aligned}\right\} \quad \text{in } I \times \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an exterior domain with Lipschitz boundary (i.e. $\Omega = \mathbb{R}^3 \setminus \overline{O}$ with $O \in C^{0,1}$ a bounded simply connected domain), $I = (0, \infty)$ is a time interval, $\varrho(t, x) : I \times \Omega \mapsto \mathbb{R}_0^+$ represents the density of the compressible fluid, $\boldsymbol{u}(t, x) : I \times \Omega \mapsto \mathbb{R}^3$ represents the velocity field, and p, a given function from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$, is the pressure. The volume force $\boldsymbol{f}$ will be assumed as potential and independent of time, i.e. $\boldsymbol{f} = \nabla F$, where $F : \Omega \to \mathbb{R}$. Eq. $(1.1)_1$ expresses the balance of momentum while $(1.1)_2$, usually called the continuity equation, expresses the balance of mass. In order to close the system (1.1), we must add initial and boundary conditions as well as conditions at infinity.

* Corresponding author.
E-mail addresses: novotny@univ-tln.fr (A. Novotný), pokorny@karlin.mff.cuni.cz (M. Pokorný).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.02.044

It is well known that due to the possible existence of vacuum regions, momentum is a more appropriate quantity than velocity in this context. We prescribe

$$\begin{aligned} \varrho(0, x) &= \varrho_0(x) \\ \boldsymbol{q}(0, x) &= \boldsymbol{q}_0(x), \end{aligned} \tag{1.2}$$

where $\boldsymbol{q}$ is the linear momentum, $\boldsymbol{q}(t, x) = \varrho(t, x)\boldsymbol{u}(t, x)$. The fact that the fluid adheres at the fixed wall is expressed by

$$\boldsymbol{u}(t, x) = \boldsymbol{0} \quad \text{at } I \times \partial\Omega. \tag{1.3}$$

Finally,

$$\left.\begin{aligned} |\boldsymbol{u}(t, x)| &\to 0 \\ \varrho(t, x) &\to \varrho_\infty \end{aligned}\right\} \quad \text{as } |x| \to \infty, \tag{1.4}$$

where ϱ_∞ is a given positive constant. Note that due to $(1.4)_2$, the total mass of the fluid must be infinite. In this paper, we shall mostly deal with

$$p(s) = s^\gamma, \quad \gamma > 1 \text{ a constant;} \tag{1.5}$$

some generalizations will be mentioned at the end of the paper. We also assume that the viscosity coefficients verify

$$\mu > 0, \quad 3\lambda + 4\mu > 0$$

which implies that the operator $-\mu\Delta - (\mu + \lambda)\nabla\mathrm{div}$ with homogeneous boundary conditions (1.3) is strongly elliptic. We shall precisely specify the sense in which (1.1)–(1.5) is satisfied in Definition 1 below.

Our aim is the following. If the force is potential and independent of time, the velocity field is expected to decay to zero and the density should approach the solution to the problem

$$\nabla p(\varrho_s) = \varrho_s \nabla F \quad \text{in } \Omega. \tag{1.6}$$

This was shown in [1,2] for bounded domains, and in [3] both for bounded domains and for exterior domains with finite initial mass, i.e. $\int_\Omega \varrho(t, x)\mathrm{d}x = \int_\Omega \varrho_0(x)\mathrm{d}x = m < \infty$ (it necessarily means that $\varrho_\infty = 0$).

In our case, we expect that the spatial asymptotic behavior of the density $(1.4)_2$ will be conserved. We thus add to (1.6)

$$\varrho_s(x) \to \varrho_\infty \quad \text{as } |x| \to \infty. \tag{1.7}$$

The sense in which (1.7) holds will be specified below.

The organization of the paper is the following. In the next section, we recall known results on the solvability of (1.1)–(1.5) and we state our main result — Theorem 2. Section 3 is devoted to proofs of global estimates of the solution which will allow us to speak about the limit density — a candidate solution to (1.6) and (1.7). Section 4 contains improved estimates of the density and a certain compensated compactness argument which allows us to pass to the limit in (1.1). Section 5 deals with the existence and the uniqueness of solutions to the rest-state problem (1.6) and (1.7). In the last section, we finish the proof of Theorem 2 and sketch the proofs for more general pressure functions than (1.5).

Throughout the paper, we adopt the standard notation for Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{k,p}(\Omega)$, homogeneous Sobolev spaces $D^{1,p}(\Omega)$ (or $W_0^{k,p}(\Omega)$ and $D_0^{1,p}(\Omega)$ for the corresponding spaces with zero traces), spaces of continuous functions $C(I)$ and certain classes of Orlicz spaces $L_q^p(\Omega)$. The norms are denoted by $\|\cdot\|_{0,p,\Omega}$ and $\|\cdot\|_{k,p,\Omega}$ for Lebesgue and Sobolev spaces, respectively. If there is no danger of confusion, domain Ω is not explicitly mentioned. The Bochner spaces on an interval I with values in X are denoted by $L^p(I; X)$ and $C(I; X)$, respectively. Vector-valued functions are printed in bold-face. The generic constants are denoted by C and their value may vary even in the same formula, or in the same line.

2. Weak renormalized bounded energy solutions

Before defining precisely what we mean under weak solutions to our system (1.1)–(1.5), we shall make several rather formal points.

Take $b(\cdot) \in C^1([0,\infty))$ and multiply $(1.1)_2$ by $b'(\varrho)$. The equality formally reads

$$\frac{\partial}{\partial t} b(\varrho) + \operatorname{div}(b(\varrho)\boldsymbol{u}) + (\varrho b'(\varrho) - b(\varrho))\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } I \times \Omega. \tag{2.1}$$

This is the well known renormalized form of the continuity equation. It is equivalent to the continuity equation provided the couple $(\varrho, \boldsymbol{u})$ is regular. For weak solutions, we shall require (2.1) to hold for all functions $b(\cdot)$ such that

$$\begin{aligned} & b(\cdot) \in C^0[0,\infty) \bigcap C^1(0,\infty) \\ & |b'(s)| \le C s^{-\lambda_0}, \quad s \in (0,1], \lambda_0 < 1 \\ & |b'(s)| \le C s^{\lambda_1}, \quad s \ge 1, -1 < \lambda_1 \le \frac{5\gamma}{6} - \frac{3}{2}. \end{aligned} \tag{2.2}$$

Next, let us multiply $(1.1)_1$ by $\boldsymbol{u}$, integrate over Ω, and apply formally Green's theorem. Using appropriately Eq. $(1.1)_2$, we get the energy equality in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 + P(\varrho) \right) \mathrm{d}x + \mu \int_\Omega |\nabla \boldsymbol{u}|^2 \mathrm{d}x + (\mu + \lambda) \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 \mathrm{d}x = \int_\Omega \varrho \boldsymbol{u} \cdot \nabla F \mathrm{d}x, \tag{2.3}$$

where

$$P(\varrho) = \frac{1}{\gamma - 1} \left(\varrho^\gamma - \gamma \varrho \varrho_\infty^{\gamma-1} + (\gamma - 1) \varrho_\infty^\gamma \right). \tag{2.4}$$

Due to the lower weak semicontinuity of the convex functional $v \mapsto \int_\Omega v^2 \mathrm{d}x$ in $L^2(\Omega)$, for weak solutions, one expects inequality rather than equality. We shall equally see later on that the integrated form (over time) of (2.3) is more appropriate for our purpose. We shall therefore postulate that the weak solution satisfies

$$\mathcal{E}(\varrho, \boldsymbol{q})(t) + \mu \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2 \mathrm{d}x \mathrm{d}s + (\mu + \lambda) \int_0^t \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 \mathrm{d}x \mathrm{d}s \le \int_0^t \int_\Omega \varrho \boldsymbol{u} \cdot \nabla F \mathrm{d}x \mathrm{d}s + \mathcal{E}(\varrho_0, \boldsymbol{q}_0), \tag{2.5}$$

where

$$\mathcal{E}(\varrho, \boldsymbol{q}) = \int_\Omega \left(\frac{1}{2} \frac{|\boldsymbol{q}|^2}{\varrho} 1_{\{\varrho > 0\}} + P(\varrho) \right) \mathrm{d}x. \tag{2.6}$$

Finally, we require that the initial data satisfy

$$\begin{aligned} & \varrho_0 \in L^1_{\mathrm{loc}}(\overline{\Omega}), \qquad P(\varrho_0) \in L^1(\Omega), \quad \varrho_0 \ge 0 \text{ a.e. in } \Omega \\ & \boldsymbol{q}_0 \in (L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\overline{\Omega}))^3, \qquad \frac{|\boldsymbol{q}_0|^2}{\varrho_0} 1_{\{\varrho_0 > 0\}} \in L^1(\Omega), \qquad \boldsymbol{q}_0 = \boldsymbol{0} \quad \text{in } \{x \in \Omega; \varrho_0 = 0\}, \end{aligned} \tag{2.7}$$

and that the volume force verifies

$$\nabla F \in L^1(\Omega) \bigcap L^\infty(\Omega). \tag{2.8}$$

Now, we are in a position to define weak solutions to problem (1.1)–(1.5) (cf. Definition 7.3 in [4]).

Definition 1. A couple $(\varrho, \boldsymbol{u})$ is called a renormalized bounded energy weak solution to (1.1)–(1.5) provided

(i) $\varrho \in L^s_{\mathrm{loc}}(I; L^s_{\mathrm{loc}}(\Omega))$ for some $s \ge \gamma$, $P(\varrho) \in L^\infty_{\mathrm{loc}}(I; L^1(\Omega))$, $\varrho \ge 0$ a.e. in $I \times \Omega$
(ii) $\boldsymbol{u} \in L^2_{\mathrm{loc}}(I; (D^{1,2}_0(\Omega))^3)$, $\varrho |\boldsymbol{u}|^2 \in L^\infty_{\mathrm{loc}}(I; L^1(\Omega))$
(iii) Eq. $(1.1)_2$ holds in $\mathcal{D}'(I \times \mathbb{R}^3)$ provided $(\varrho, \boldsymbol{u})$ is extended by zero outside of Ω
(iv) Eq. (2.1) holds in $\mathcal{D}'(I \times \mathbb{R}^3)$ for any $b(\cdot)$ satisfying (2.2) provided $(\varrho, \boldsymbol{u})$ is extended by zero outside of Ω
(v) Eq. $(1.1)_1$ holds in $(\mathcal{D}'(I \times \Omega))^3$
(vi) $\lim_{t \to 0^+} \int_\Omega \varrho(t, \cdot) \psi \mathrm{d}x = \int_\Omega \varrho_0 \psi \mathrm{d}x$, $\forall \psi \in \mathcal{D}(\Omega)$
(vii) $\lim_{t \to 0^+} \int_\Omega (\varrho \boldsymbol{u})(t, \cdot) \cdot \boldsymbol{\varphi} \mathrm{d}x = \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\varphi} \mathrm{d}x$, $\forall \boldsymbol{\varphi} \in (\mathcal{D}(\Omega))^3$
(viii) inequality (2.5) holds a.e. in I.

Remark 1. Note that $P(\varrho) \in L^\infty(I; L^1(\Omega))$ can be expressed in a more suitable form using Orlicz spaces, cf. [5]. One can easily check that

$$P(\varrho) \sim (\varrho - \varrho_\infty)^2 \quad \text{if } |\varrho - \varrho_\infty| \le 1$$
$$P(\varrho) \sim (\varrho - \varrho_\infty)^\gamma \quad \text{if } |\varrho - \varrho_\infty| > 1.$$

We put for $1 < p, q < \infty$

$$L^p_q(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega); \int_\Omega \Phi(|u|)\mathrm{d}x < \infty \right\}$$

with the corresponding norm

$$\|u\|_{L^p_q(\Omega)} = \sup\left\{ \int_\Omega uv\mathrm{d}x; \int_\Omega \Psi(|v|)\mathrm{d}x \le 1 \right\},$$

where Φ and Ψ are complementary Young's functions such that $\Phi(s) \sim s^q$ for $s \in (0, \delta_1)$ and $\Phi(s) \sim s^p$ for $s \in (\delta_2, \infty)$ for some $0 < \delta_1 < \delta_2$. It is an easy matter to see that one can choose Φ such that it satisfies the Δ_2 condition. One can also easily verify that $L^p_q(\Omega)$ does not depend on the choice of δ_1 and δ_2. The space is a separable reflexive Banach space; its dual is isometrically isomorphic to $L^{p'}_{q'}(\Omega)$. Moreover, $\mathcal{D}(\Omega)$ is a dense subset of $L^p_q(\Omega)$ and

$$\|u 1_{\{|u|<\delta\}}\|_{0,q,\Omega} + \|u 1_{\{|u|\ge\delta\}}\|_{0,p,\Omega}, \qquad \delta > 0$$

are equivalent norms in $L^p_q(\Omega)$, see e.g. [6]. Note that $L^p_q(\Omega) = L^p(\Omega) \cap L^q(\Omega)$ provided $p \ge q$. Thus $P(\varrho) \in L^\infty(I; L^1(\Omega))$ can be equivalently expressed by the fact that $\varrho \in L^\infty(I; L^\gamma_2(\Omega))$.

The first existence theorem for weak solutions to system (1.1) is due to Lions [5] for values $\gamma \ge \frac{9}{5}$. Later on, it was generalized in the spirit of Feireisl's result [7] up to $\gamma > \frac{3}{2}$. The following existence theorem is taken over from [4] (Theorem 7.15 and bibliographic remarks in Chapter 7). Here, and in what follows, $C(\overline{I}; X_w)$ denotes the space of all functions continuous on $\overline{I}$ in the weak topology of X.

Theorem 1. *Let $\Omega \in C^{0,1}$, and let the data satisfy (2.7) and (2.8). Let $\gamma > \frac{3}{2}$. Then there exists a renormalized bounded energy weak solution to (1.1)–(1.5) in the sense of Definition 1. Moreover, we have*

(i) $\varrho \in L^{\frac{5\gamma-3}{3}}_{\text{loc}}((0,\infty) \times \Omega)$, $\varrho \in C([0,\infty); L^\gamma(\Omega')_w) \cap C([0,\infty); L^p(\Omega'))$, $1 \le p < \gamma$, *where Ω' is any bounded subdomain of Ω*

(ii) $\varrho\boldsymbol{u} \in C([0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\Omega')_w)^3)$, $\varrho|\boldsymbol{u}|^2 \in L^2_{\text{loc}}((0,\infty); L^{\frac{6\gamma}{4\gamma+3}}(\Omega'))$

(iii) *the energy $\mathcal{E}(\varrho, \boldsymbol{q})$ defined in (2.6) is lower semicontinuous on $[0,\infty)$ and the integral form of the energy inequality (2.5) holds a.e. in I.*

Note that instead of (2.5), one could require a differential form of the energy inequality, analogous to (2.3). But for exterior domains, the existence of such solutions, called finite energy solutions, is not known. It is connected with the missing control on the approximate sequence of densities ϱ_n outside large balls, and thus it is not clear whether $\varrho_n - \varrho_\infty \to \varrho - \varrho_\infty$ in $L^p(I; L^\gamma_2(\Omega))$ for some $p \ge 1$. We shall meet similar problems later on.

Before formulating our main result, let us summarize the conditions which we additionally impose on the potential F in order to be able to study the asymptotic behavior of solutions guaranteed by Theorem 1. These conditions imply the uniqueness of the solution to (1.6) and (1.7), see Section 5:

$$\begin{aligned}
&F \in L^2(\Omega) \bigcap C^{0,1}_B(\overline{\Omega}), \qquad F(x) \to 0 \quad \text{as } |x| \to \infty;\\
&\left\{ x \in \Omega; F(x) > -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1} \right\} \text{ is connected;}\\
&\text{any connected component } \widetilde{\Omega}_i \text{ of } \left\{ x \in \Omega; F(x) < -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1} \right\}\\
&\text{is such that the sets } \{x \in \widetilde{\Omega}_i; F(x) > k\}\\
&\text{are connected for all } k < -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}.
\end{aligned} \tag{2.9}$$

Recall that $C_B^{0,1}(\overline{\Omega})$ denotes the sets of all Lipschitz continuous functions up to the boundary which are bounded in Ω. The main achievement of this paper is the following result.

Theorem 2. *Let $(\varrho, \boldsymbol{u})$ be a renormalized bounded energy weak solution to (1.1)–(1.5) such that the density $\varrho \in C([0,\infty); L^\gamma(\Omega')_w)$ and moreover the momentum $\varrho\boldsymbol{u} \in C([0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\Omega')_w)^3)$ for $\Omega' \subset \Omega$ any bounded domain. Let $\gamma > \frac{3}{2}$ and $\Omega \in C^{0,1}$. Let F satisfy (2.8) and (2.9). Then, as $t \to \infty$, we have*

$$\varrho - \varrho_\infty \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L_2^\gamma(\Omega)$$
$$\varrho \to \varrho_s \quad \text{in } L^p(\Omega'),\ 1 \le p < \gamma,\ \Omega' \subset \Omega \text{ any bounded domain}$$
$$\varrho\boldsymbol{u} \rightharpoonup \mathbf{0} \quad \text{in } (L_2^{\frac{2\gamma}{\gamma+1}}(\Omega))^3,$$

where ϱ_s is the unique solution to

$$\nabla(\varrho_s^\gamma) = \varrho_s \nabla F \quad \text{in } \Omega$$
$$\varrho_s - \varrho_\infty \in L_2^\gamma(\Omega)$$
$$\varrho_s \ge 0 \quad \text{a.e.in } \Omega.$$

If we compare our result with the results of [3,1] or [2], we see that we are able to prove only local convergence of the density, and we have almost no reasonable information on the convergence of momentum and kinetic energy. This is connected with the missing control on the density near infinity, which does not allow us to pass to the limit in the energy inequality. In the case $\varrho_\infty = 0$ and $\int_\Omega \varrho_0 \mathrm{d}x$ finite, treated in [3], the missing information was provided by the L^1-estimate of the density.

3. Energy inequality and its consequences

Let $(\varrho, \boldsymbol{u})$ be a renormalized bounded energy weak solution to (1.1)–(1.5) in the sense of Definition 1. Further (cf. Theorem 2), let

$$\left.\begin{aligned} &\varrho \in C([0,\infty); L^\gamma(\Omega')_w) \\ &\varrho\boldsymbol{u} \in C([0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\Omega')_w)^3), \end{aligned}\right. \quad \Omega' \subset \Omega \text{ any bounded domain.} \tag{3.1}$$

We would like to show that both ϱ and $\varrho\boldsymbol{u}$ are continuous on $[0,\infty)$ in weak topologies of $L_2^\gamma(\Omega)$ and $L_2^{\frac{2\gamma}{\gamma+1}}$, respectively.

Lemma 1. *Let ϱ and $\varrho\boldsymbol{u}$ satisfy (3.1) and let $P(\varrho)$ and $\varrho|\boldsymbol{u}|^2 \in L^\infty_{\mathrm{loc}}(I; L^1(\Omega))$. Then $\varrho - \varrho_\infty \in C([0,\infty); L_2^\gamma(\Omega)_w)$ and $\varrho\boldsymbol{u} \in C([0,\infty); (L_2^{\frac{2\gamma}{\gamma+1}}(\Omega)_w)^3)$.*

Proof. Evidently, it is possible to assume that $\varrho(t,\cdot)$ is defined for all $t \in [0,\infty)$ as an element of $L_2^\gamma(\Omega)$,

$$\|\varrho(t) - \varrho_\infty\|_{L_2^\gamma(\Omega)} \le \|\varrho - \varrho_\infty\|_{L^\infty(I; L_2^\gamma(\Omega))} \quad \forall t \in [0,\infty)$$

and $\varrho \in C([0,\infty); (L^\gamma(\Omega')_w))$ for any Ω' a bounded subdomain of Ω. Let us fix an arbitrary $\varphi \in L_2^{\gamma'}(\Omega)$ and $\varepsilon > 0$. Choose any $t_0 \in [0,\infty)$. Then for any $t \in [0,\infty)$ we have ($\Omega_R = \Omega \cap B_R$, $\Omega^R = \Omega \setminus \overline{\Omega_R}$)

$$\begin{aligned} &\left| \int_\Omega (\varrho(t,\cdot) - \varrho_\infty)\varphi \mathrm{d}x - \int_\Omega (\varrho(t_0,\cdot) - \varrho_\infty)\varphi \mathrm{d}x \right| \\ &\quad \le \left| \int_{\Omega_R} (\varrho(t,\cdot) - \varrho(t_0,\cdot))\varphi \mathrm{d}x \right| + \left| \int_{\Omega^R} (\varrho(t,\cdot) - \varrho(t_0,\cdot))\varphi \mathrm{d}x \right| \\ &\quad \le \left| \int_{\Omega_R} (\varrho(t,\cdot) - \varrho(t_0,\cdot))\varphi \mathrm{d}x \right| + 2\|\varrho\|_{L^\infty(I; L_2^\gamma(\Omega))} \|\varphi\|_{L_2^{\gamma'}(\Omega^R)}. \end{aligned} \tag{3.2}$$

First, we choose R sufficiently large so that the second term is less than $\frac{\varepsilon}{2}$. Next, having R fixed, we take δ sufficiently small so that for all $t \in \mathcal{U}_\delta(t_0) \cap [0,\infty)$, the first term is less than $\frac{\varepsilon}{2}$. As $\varphi \in L_2^{\gamma'}(\Omega)$ was arbitrary, the continuity of ϱ is proved.

The proof for the momentum is essentially the same. Just recall that

$$\int_\Omega (\varrho|\boldsymbol{u}|)^2 1_{\{|\varrho \boldsymbol{u}|\le 1\}} \mathrm{d}x = \int_\Omega (\varrho|\boldsymbol{u}|)^2 1_{\{\varrho\le\varrho_\infty+1\}\cap\{|\varrho\boldsymbol{u}|\le 1\}} \mathrm{d}x + \int_\Omega (\varrho|\boldsymbol{u}|)^2 1_{\{\varrho>\varrho_\infty+1\}\cap\{|\varrho\boldsymbol{u}|\le 1\}} \mathrm{d}x$$
$$\le (\varrho_\infty+1)\int_\Omega \varrho|\boldsymbol{u}|^2 \mathrm{d}x + |\{x\in\Omega;\varrho(t,\cdot)>\varrho_\infty+1\}|.$$

The first term is bounded, thanks to the fact that the kinetic energy belongs to $L^\infty_{\mathrm{loc}}([0,\infty);L^1(\Omega))$ and the second term is bounded, thanks to the fact that the density belongs to $L^\infty_{\mathrm{loc}}([0,\infty);L_2^\gamma(\Omega))$; indeed,

$$|\{\varrho(t,\cdot)-\varrho_\infty>1\}| \le \int_{\{\varrho(t,\cdot)-\varrho_\infty>1\}} |\varrho(t,\cdot)-\varrho_\infty|\mathrm{d}x$$
$$\le |\{\varrho(t,\cdot)-\varrho_\infty>1\}|^{\frac{\gamma-1}{\gamma}} \left(\int_{\{\varrho(t,\cdot)-\varrho_\infty>1\}} |\varrho(t,\cdot)-\varrho_\infty|^\gamma \mathrm{d}x\right)^{\frac{1}{\gamma}}.$$

One can easily check that $|\{x\in\Omega;|(\varrho|\boldsymbol{u}|)(t,\cdot)|>1\}| < K_T < \infty$ on $[0,T]$. Indeed, we can write $\{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\} = \{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\}\cap\{\varrho(t,\cdot)-\varrho_\infty>1\}\cup\{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\}\cap\{\varrho(t,\cdot)-\varrho_\infty\le 1\}$ and

$$|\{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\}\cap\{\varrho(t,\cdot)-\varrho_\infty\le 1\}| \le \int_{\{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\}\cap\{\varrho(t,\cdot)-\varrho_\infty\le 1\}} \sqrt{\varrho_\infty+1}\sqrt{\varrho(t,\cdot)}|\boldsymbol{u}(t,\cdot)|\mathrm{d}x$$
$$\le \sqrt{\varrho_\infty+1}\|(\varrho|\boldsymbol{u}|)(t,\cdot)\|_1^{\frac{1}{2}} |\{(\varrho|\boldsymbol{u}|)(t,\cdot)>1\}\cap\{\varrho(t,\cdot)-\varrho_\infty\le 1\}|^{\frac{1}{2}}.$$

Thus

$$\int_\Omega (\varrho|\boldsymbol{u}|)^{\frac{2\gamma}{\gamma+1}} 1_{\{\varrho|\boldsymbol{u}|>1\}} \mathrm{d}x = \int_\Omega (\varrho|\boldsymbol{u}|)^{\frac{2\gamma}{\gamma+1}} 1_{\{\varrho\le\varrho_\infty+1\}\cap\{\varrho|\boldsymbol{u}|>1\}} \mathrm{d}x + \int_\Omega (\varrho|\boldsymbol{u}|)^{\frac{2\gamma}{\gamma+1}} 1_{\{\varrho>\varrho_\infty+1\}\cap\{\varrho|\boldsymbol{u}|>1\}} \mathrm{d}x$$
$$\le (C + \|\varrho-\varrho_\infty\|^{\frac{\gamma}{\gamma+1}}_{L^\infty(I;L_2^\gamma(\Omega))})\|\varrho|\boldsymbol{u}|^2\|^{\frac{\gamma}{\gamma+1}}_{L^\infty(I:L^1(\Omega))},$$

where the constant $C = C(\varrho_\infty, K_T, \gamma)$. We have shown that $\varrho\boldsymbol{u}$ belongs to $L^\infty_{\mathrm{loc}}(I;(L_2^{\frac{2\gamma}{\gamma+1}}(\Omega))^3)$ and we can proceed as in (3.2) to show that in fact, the momentum also belongs to $C([0,\infty);(L_2^{\frac{2\gamma}{\gamma+1}}(\Omega)_w)^3)$. □

Using the continuity of the density, we are able to transform energy inequality (2.5) into a more useful form. We have (at this moment only formally):

$$\int_0^t\int_\Omega (\varrho\boldsymbol{u})\cdot\nabla F \mathrm{d}x\mathrm{d}s = -\int_0^t\int_\Omega \mathrm{div}(\varrho\boldsymbol{u})F\mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega \frac{\partial\varrho}{\partial t}F\mathrm{d}x\mathrm{d}s$$
$$= \int_0^t\int_\Omega \frac{\partial}{\partial t}(\varrho-\varrho_\infty)F\mathrm{d}x\mathrm{d}s = \int_\Omega(\varrho(t,\cdot)-\varrho_\infty)F\mathrm{d}x - \int_\Omega(\varrho_0-\varrho_\infty)F\mathrm{d}x.$$

The following lemma shows that this formal calculation can be rigorously justified.

Lemma 2. *Let $F\in L^2(\Omega)$, $\nabla F\in (L^2(\Omega))^3\cap(L^6(\Omega))^3$, Ω an exterior Lipschitz domain, $\gamma>\frac{3}{2}$. Then for any $t\in(0,\infty)$:*

$$\int_0^t\int_\Omega(\varrho\boldsymbol{u})\cdot\nabla F\mathrm{d}x = \int_\Omega(\varrho(t,\cdot)-\varrho_\infty)F\mathrm{d}x - \int_\Omega(\varrho_0-\varrho_\infty)F\mathrm{d}x. \tag{3.3}$$

Proof. Thanks to the estimates of the momentum shown above, the integral on the left hand-side of (3.3) is finite and defined for any $t\in[0,\infty)$. There exists a sequence of $F_n\in C_0^\infty(\Omega)$ such that $F_n\to F$ in $L^2(\Omega)\cap L^{\max\{\gamma',2\}}(\Omega)$ and

$\nabla F_n \to \nabla F$ in $(L^2(\Omega))^3 \cap (L^6(\Omega))^3$. Take further a sequence $\eta_m \in C_0^\infty(0,t)$ such that η_m converges uniformly to η_δ and η_m' converges pointwise almost everywhere to η_δ', where

$$\eta_\delta(\tau) = \begin{cases} \dfrac{\tau}{\delta} & \tau \in [0,\delta) \\ 1 & \tau \in [\delta, t-\delta] \\ \dfrac{t-\tau}{\delta} & \tau \in (t-\delta, t]. \end{cases}$$

Moreover, let η_m' be uniformly bounded on $[0,t]$.

Using the fact that the pair $(\varrho, \boldsymbol{u})$ is a renormalized weak solution to the continuity equation, we have

$$\int_0^t \int_\Omega (\varrho \boldsymbol{u}) \cdot \nabla F_n \eta_m \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{R}^3} (\varrho \boldsymbol{u}) \cdot \nabla F_n \eta_m \mathrm{d}x\mathrm{d}s$$
$$= -\int_0^t \int_{\mathbb{R}^3} \varrho F_n \eta_m' \mathrm{d}x\mathrm{d}s = -\int_0^t \int_\Omega (\varrho - \varrho_\infty) F_n \eta_m' \mathrm{d}x\mathrm{d}s.$$

We may now pass with $m \to \infty$. We get

$$\int_0^t \int_\Omega (\varrho \boldsymbol{u}) \cdot \nabla F_n \eta_\delta \mathrm{d}x\mathrm{d}s = \frac{1}{\delta} \int_{t-\delta}^t \int_\Omega (\varrho - \varrho_\infty) F_n \mathrm{d}x\mathrm{d}s - \frac{1}{\delta} \int_0^\delta \int_\Omega (\varrho - \varrho_\infty) F_n \mathrm{d}x\mathrm{d}s.$$

Passing with $n \to \infty$ and finally with $\delta \to 0^+$, the equality yields, thanks to the continuity of $\varrho - \varrho_\infty$ in $L_2^\gamma(\Omega)_w$, the assertion of the lemma. $\square$

Applying Lemma 2 to the energy inequality, we get

$$\int_\Omega \left(\frac{1}{2} \varrho(t,\cdot) |\boldsymbol{u}(t,\cdot)|^2 + P(\varrho(t,\cdot)) - (\varrho(t,\cdot) - \varrho_\infty) F \right) \mathrm{d}x$$
$$+ \mu \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2 \mathrm{d}x\mathrm{d}s + (\mu + \lambda) \int_0^t \int_\Omega |\mathrm{div}\, \boldsymbol{u}|^2 \mathrm{d}x\mathrm{d}s$$
$$\leq \int_\Omega \left(\frac{1}{2} \frac{|\boldsymbol{q}_0|^2}{\varrho_0} 1_{\{\varrho_0 > 0\}} + P(\varrho_0) - (\varrho_0 - \varrho_\infty) F \right) \mathrm{d}x. \tag{3.4}$$

Evidently, by Young's inequality, we can estimate the term $\int_\Omega \varrho F \mathrm{d}x$ by $\frac{1}{2} \int_\Omega P(\varrho) \mathrm{d}x$ and a term depending on F; thus (3.4) yields the following estimates

$$\begin{aligned} &\varrho |\boldsymbol{u}|^2 \in L^\infty((0,\infty); L^1(\Omega)) \\ &P(\varrho) \in L^\infty((0,\infty); L^1(\Omega)) \quad (\text{i.e. } \varrho - \varrho_\infty \in L^\infty((0,\infty); L_2^\gamma(\Omega))) \\ &\nabla \boldsymbol{u} \in L^2((0,\infty); (L^2(\Omega))^9) \\ &\varrho \boldsymbol{u} \in L^\infty((0,\infty); (L_2^{\frac{2\gamma}{\gamma+1}}(\Omega))^3). \end{aligned} \tag{3.5}$$

Let us take an arbitrary sequence $\tau_n \to \infty$ and denote

$$\left. \begin{aligned} \varrho_n(t,\cdot) &= \varrho(t + \tau_n, \cdot) \\ \boldsymbol{u}_n(t,\cdot) &= \boldsymbol{u}(t + \tau_n, \cdot) \end{aligned} \right\} \quad t \in (0,3).$$

Evidently, the pairs $(\varrho_n, \boldsymbol{u}_n)$ solve (1.1) $\forall n \in \mathbb{N}$. Moreover, due to (3.5), $\varrho_n - \varrho_\infty$ is bounded in $L^\infty((0,3); L_2^\gamma(\Omega))$ and $\nabla \boldsymbol{u}_n \to \boldsymbol{0}$ in $(L^2((0,3) \times \Omega))^9$, $\boldsymbol{u}_n \to \boldsymbol{0}$ in $(L^2((0,3); L^6(\Omega)^3))$. Therefore there exists a subsequence ϱ_{n_k} such that

$$\varrho_{n_k} - \varrho_\infty \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L^r((0,3); L_2^\gamma(\Omega)), 1 \leq r < \infty.$$

We may now easily pass to the limit in the weak formulation of the continuity equation to see that ϱ_s must be independent of time. However, in order to pass to the limit in the momentum equation $(1.1)_1$, we need a better estimate of the density. This will be the aim of the following section.

4. Improved estimates of the density

The aim of this section is to prove estimates of the type

$$\int_{\frac{1}{2}}^{\frac{5}{2}} \int_{\Omega} \varrho_n^{\gamma+\theta} \psi \mathrm{d}x\mathrm{d}t \le C(\psi),$$

where $\theta = \frac{2}{3}\gamma - 1$ for γ near $\frac{3}{2}$ and ψ is an arbitrary cut-off function such that $\psi = 1$ in Ω_{R-1} for some $R > 1$, $\operatorname{supp} \psi \subset \Omega_R$. Such an estimate is a relatively standard tool in the theory of compressible Navier–Stokes equations. We adopt the technique used in [4], Section 7.9.5. For another approach leading to the same result, see [5] or [8]. The main idea is to take as a test function for $(1.1)_2$ the function

$$\xi(t,x) = \eta(t)\mathcal{B}_{\Omega}(\varrho_n^{\theta}\psi)(t,x), \tag{4.1}$$

where $\eta \in C_0^{\infty}(0,3)$, $\eta \equiv 1$ in $[\frac{1}{2}, \frac{5}{2}]$ and $\mathcal{B}_{\Omega}$ is the so-called Bogovskii operator, i.e. the solution operator to

$$\begin{aligned} &\operatorname{div} \boldsymbol{w} = f \quad \text{in } \Omega \\ &\boldsymbol{w} = \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Note that Ω is unbounded, and thus the compatibility condition $\int_{\Omega} f \mathrm{d}x = 0$ is not necessary. It is well known (see e.g. [4]) that

$$\|\nabla \mathcal{B}_{\Omega}(f)\|_p \le C(p)\|f\|_p, \quad 1 < p < \infty.$$

However, because of the lack of regularity of the time derivative of the linear momentum, we cannot use ξ defined in (4.1) directly. Thus we take

$$b_k(s) = \begin{cases} s^{\theta} & 0 < s \le k \\ k^{\theta} & s > k. \end{cases}$$

Using a standard regularization technique (cf. the proof of Lemma 2), one may show that $b_k(\cdot)$ is a suitable function in the renormalized continuity equation (2.1).

Further, denote by $S_\alpha(g)(s)$ the mollification of a function g in time, i.e.

$$S_\alpha(g)(s) = \frac{1}{\alpha}\int_0^3 \omega_0\left(\frac{s-\tau}{\alpha}\right) g(\tau)\mathrm{d}\tau, \quad \omega_0(\cdot) \in C_0^{\infty}(-1,1).$$

It is well known that for any $I' \subset \overline{I'} \subset (0,3)$ and α sufficiently small, we have

$$\begin{aligned} &\|S_\alpha g\|_{L^p(I')} \le C\|g\|_{L^p((0,3))}, \quad 1 \le p \le \infty \\ &\lim_{\alpha\to 0^+} \|S_\alpha g - g\|_{L^p(I')} = 0, \quad 1 \le p < \infty. \end{aligned}$$

Moreover, $S_\alpha g \in C^{\infty}(\overline{I'})$ if $g \in L^1_{\mathrm{loc}}((0,3))$. Thus, we have for any $I' \subset \overline{I'} \subset (0,3)$ and $\alpha < \alpha_0(I')$ with α_0 sufficiently small

$$\frac{\partial}{\partial t} S_\alpha(b_k(\varrho_n)) + \operatorname{div} S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n) + S_\alpha\{[\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div} \boldsymbol{u}_n\} = 0$$

in $\mathcal{D}'(I' \times \mathbb{R}^3)$. Here, $(b_k')_+(\cdot)$ denotes the derivative from the right.

Note that

$$\begin{aligned} &S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n) \in C^{\infty}(I'; L^6(\mathbb{R}^3)) \\ &S_\alpha(b_k(\varrho_n)) \in C^{\infty}(I'; L^{\infty}(\mathbb{R}^3)) \\ &S_\alpha\{[\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div} \boldsymbol{u}_n\} \in C^{\infty}(I'; L^2(\mathbb{R}^3)) \\ &\operatorname{div} S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n) \in C^{\infty}(I'; L^2_{\mathrm{loc}}(\mathbb{R}^3)) \end{aligned}$$

and the normal trace of $S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)$ is zero in $(W^{\frac{1}{2},2}(\partial\Omega))^*$. Therefore, thanks to the standard density argument, we can use as a test function in $(1.1)_2$

$$\boldsymbol{\varphi}(t,x) = \eta(t)\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)(t,x) \tag{4.2}$$

with η and ψ defined above. Let us compute first

$$\begin{aligned}\frac{\partial \boldsymbol{\varphi}}{\partial t} &= \eta'\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi) + \eta\mathcal{B}_\Omega\left(\frac{\partial}{\partial t}S_\alpha(b_k(\varrho_n))\psi\right)\\ &= \eta'\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi) - \eta\mathcal{B}_\Omega(\operatorname{div}[S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\psi])\\ &\quad + \eta\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi) - \eta\mathcal{B}_\Omega(S_\alpha\{[\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div}\boldsymbol{u}_n\}\psi).\end{aligned}$$

Thus we have ($I := (0,3)$)

$$\begin{aligned}\int_I\int_\Omega \eta\varrho_n^\gamma S_\alpha(b_k(\varrho_n))\psi\,\mathrm{d}x\mathrm{d}t &= \mu\int_I\int_\Omega \nabla\boldsymbol{u}_n : \nabla\boldsymbol{\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad + (\mu+\lambda)\int_I\int_\Omega \operatorname{div}\boldsymbol{u}_n\operatorname{div}\boldsymbol{\varphi}\mathrm{d}x\mathrm{d}t - \int_I\int_\Omega \varrho_n(\boldsymbol{u}_n\otimes\boldsymbol{u}_n) : \nabla\boldsymbol{\varphi}\mathrm{d}x\mathrm{d}t\\ &\quad + \int_I\int_\Omega \eta\mathcal{B}_\Omega(\operatorname{div}[S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\psi])\cdot\boldsymbol{u}_n\varrho_n\mathrm{d}x\mathrm{d}t\\ &\quad - \int_I\int_\Omega \eta'\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)\cdot\varrho_n\boldsymbol{u}_n\mathrm{d}x\mathrm{d}t - \int_I\int_\Omega \varrho_n\boldsymbol{\varphi}\cdot\nabla F\mathrm{d}x\mathrm{d}t\\ &\quad - \int_I\int_\Omega \eta\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)\cdot\varrho_n\boldsymbol{u}_n\mathrm{d}x\mathrm{d}t\\ &\quad + \int_I\int_\Omega \eta\mathcal{B}_\Omega(S_\alpha\{[\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div}\boldsymbol{u}_n\}\psi)\cdot\varrho_n\boldsymbol{u}_n\mathrm{d}x\mathrm{d}t := \sum_{i=1}^8 \mathcal{J}_i.\end{aligned}$$

We estimate each of $\mathcal{J}_i$, $i = 1,\dots,8$, independently of k, α, and n, which will allow us to pass with $\alpha\to 0^+$ and $k\to\infty$ later on. We have

$$\begin{aligned}|\mathcal{J}_1| + |\mathcal{J}_2| &\le C\|\nabla\boldsymbol{u}_n\|_{L^2(I;L^2(\Omega))}\|\nabla\boldsymbol{\varphi}\|_{L^2(I;L^2(\Omega))} \le C\|S_\alpha(b_k(\varrho_n))\psi\|_{L^2(I;L^2(\Omega))}\\ &\le C\|\varrho_n^\theta\|_{L^2(I;L^2(\Omega_R))} \le C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \frac{\gamma}{2}$.

$$\begin{aligned}|\mathcal{J}_3| &\le \int_I\int_\Omega \varrho_n|\boldsymbol{u}_n|^2|\nabla\boldsymbol{\varphi}|\mathrm{d}x\mathrm{d}t \le \|\boldsymbol{u}_n\|^2_{L^2(I;L^6(\Omega))}\\ &\quad\times\left(\|\varrho_n - \varrho_\infty\|_{L^\infty(I;L_2^\gamma(\Omega))}\|\nabla\boldsymbol{\varphi}\|_{L^\infty(I;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} + C(\varrho_\infty)\|\nabla\boldsymbol{\varphi}\|_{L^\infty(I;L^{\frac{3}{2}}(\Omega))}\right)\\ &\le C\left(\|S_\alpha(b_k(\varrho_n))\psi\|_{L^\infty(I;L^{\frac{3\gamma}{2\gamma-3}}(\Omega))} + \|S_\alpha(b_k(\varrho_n))\psi\|_{L^\infty(I;L^{\frac{3}{2}}(\Omega))}\right)\\ &\le C\left(\|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{3\gamma}{2\gamma-3}}(\Omega_R))} + \|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{3}{2}}(\Omega_R))}\right) \le C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \frac{2}{3}\gamma - 1$. Completely analogously, we may also estimate $\mathcal{J}_4$ (recall that we have $\|\mathcal{B}_\Omega(\operatorname{div}[S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\psi])\|_p \le C(p)\|S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\psi\|_p$, $1 < p < \infty$).

$$\begin{aligned}\mathcal{J}_5 &\le C\|\varrho_n - \varrho_\infty\|^{\frac{1}{2}}_{L^\infty(I;L_2^\gamma(\Omega))}\|\varrho_n|\boldsymbol{u}_n|^2\|^{\frac{1}{2}}_{L^\infty(I;L^1(\Omega))}\|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)\|_{L^\infty(I;L^{\frac{2\gamma}{\gamma-1}}(\Omega))}\\ &\quad + C(\varrho_\infty)\|\varrho_n|\boldsymbol{u}_n|^2\|^{\frac{1}{2}}_{L^\infty(I;L^1(\Omega))}\|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)\|_{L^\infty(I;L^2(\Omega))}\\ &\le C\left(\|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)\|_{L^\infty(I;L^{\frac{6\gamma}{5\gamma-3}}(\Omega))} + \|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n))\psi)\|_{L^\infty(I;L^{\frac{6}{5}}(\Omega))}\right)\\ &\le C\left(\|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{6\gamma}{5\gamma-3}}(\Omega_R))} + \|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{6}{5}}(\Omega_R))}\right) \le C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \frac{5}{6}\gamma - \frac{1}{2}$.

$$\begin{aligned}\mathcal{J}_6 &\le C(\varrho_\infty)\|\nabla F\|_2\|\boldsymbol{\varphi}\|_{L^\infty(I;L^2(\Omega))} + \|\nabla F\|_\infty \|\varrho_n - \varrho_\infty\|_{L^\infty(I;L^\gamma_2(\Omega))}\|\boldsymbol{\varphi}\|_{L^\infty(I;L_2^{\gamma'}(\Omega))}\\
&\le C\left(\|\nabla\boldsymbol{\varphi}\|_{L^\infty(I;L^{\frac{6}{5}}(\Omega))} + \|\nabla\boldsymbol{\varphi}\|_{L^\infty(I;L^{\max\{\frac{6}{5},\frac{3\gamma}{4\gamma-3}\}}(\Omega))}\right)\\
&\le C\left(\|S_\alpha(b_k(\varrho_n))\|_{L^\infty(I;L^{\frac{6}{5}}(\Omega_R))} + \|S_\alpha(b_k(\varrho_n))\|_{L^\infty(I;L^{\frac{3\gamma}{4\gamma-3}}(\Omega_R))}\right)\\
&\le C\left(\|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{6}{5}}(\Omega_R))} + \|\varrho_n^\theta\|_{L^\infty(I;L^{\frac{3\gamma}{4\gamma-3}}(\Omega_R))}\right) \le C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \min\{\frac{5}{6}\gamma, \frac{4}{3}\gamma - 1\}$.

$$\begin{aligned}\mathcal{J}_7 \le{}& \|\varrho_n - \varrho_\infty\|^{\frac{1}{2}}_{L^\infty(I;L_2^\gamma(\Omega))}\|\boldsymbol{u}_n\|_{L^2(I;L^6(\Omega))}\\
&\times \|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)1_{\varrho_n>\varrho_\infty+1}\|_{L^2(I;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))}\\
&+ C(\varrho_\infty)\|\boldsymbol{u}_n\|_{L^2(I;L^6(\Omega))}\|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)1_{\varrho_n>\varrho_\infty+1}\|_{L^2(I;L^{\frac{6}{5}}(\Omega))}\\
&+ C(\varrho_\infty)\|\varrho_n|\boldsymbol{u}_n|^2\|^{\frac{1}{4}+\delta_1}_{L^\infty(I;L^1(\Omega))} + \|\boldsymbol{u}_n\|^{\frac{1}{2}-\delta_2}_{L^2(I;L^6(\Omega))}\\
&\times \|\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)\|_{L^{\frac{4}{3}}(I;L^{\frac{3}{2}+\delta_3}(\Omega))},\end{aligned}$$

where δ_1 can be taken arbitrarily small and, independent of it, so are also δ_2 and δ_3. As the measure of the set $\{x \in \Omega; \varrho_n(x) > \varrho_\infty + 1\}$ is bounded uniformly with respect to n, and we get

$$\begin{aligned}\mathcal{J}_7 \le{}& C\Big(\|\nabla\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)\|_{L^2(I;L^{\max\{\frac{6\gamma}{7\gamma-6},1+\tilde\delta\}}(\Omega_R))} + \|\nabla\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)\|_{L^2(I;L^{1+\tilde\delta}(\Omega_R))}\\
&+ \|\nabla\mathcal{B}_\Omega(S_\alpha(b_k(\varrho_n)\boldsymbol{u}_n)\cdot\nabla\psi)\|_{L^{\frac{4}{3}}(I;L^{1+\tilde\delta}(\Omega_R))}\Big)\\
\le{}& C\|\nabla\boldsymbol{u}_n\|_{L^2(I;L^2(\Omega))}\left(\|\varrho_n^\theta\|_{L^4(I;L^{\frac{6}{5}+\tilde\delta_1}(\Omega_R))} + \|\varrho_n^\theta\|_{L^\infty(I;L^{\max\{\frac{\gamma}{\gamma-1},\frac{6}{5}+\tilde\delta-1\}}(\Omega_R))}\right)\\
\le{}& C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \min\{\gamma - 1, \frac{5}{6}\gamma - \delta\}$ for $\delta > 0$, arbitrarily small.

Analogously we estimate the last term and get

$$\begin{aligned}\mathcal{J}_8 \le{}& C\Big(\|\nabla\mathcal{B}_\Omega(S_\alpha([\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div}\boldsymbol{u}_n)\psi)\|_{L^2(I;L^{\max\{1+\tilde\delta,\frac{6\gamma}{7\gamma-6}\}}(\Omega))}\\
&+ \|\nabla\mathcal{B}_\Omega(S_\alpha([\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div}\boldsymbol{u}_n)\psi)\|_{L^2(I;L^{1+\tilde\delta}(\Omega))}\\
&+ \|\nabla\mathcal{B}_\Omega(S_\alpha([\varrho_n(b_k')_+(\varrho_n) - b_k(\varrho_n)]\operatorname{div}\boldsymbol{u}_n)\psi)1_{\{\varrho>\varrho_\infty+1\}}\|_{L^{\frac{4}{3}}(I;L^{1+\tilde\delta}(\Omega))}\Big)\\
\le{}& C\|\nabla\boldsymbol{u}_n\|_{L^2(I;L^2(\Omega))}\left(\|\varrho_n^\theta\|_{L^\infty(I;L^{\max\{2+\tilde\delta_1,\frac{3\gamma}{2\gamma-3}\}}(\Omega_R))} + \|\varrho_n^\theta\|_{L^4(I;L^{2+\tilde\delta_1}(\Omega_R))}\right)\\
\le{}& C\|\varrho_n\|^\theta_{L^\infty(I;L^\gamma(\Omega_R))},\end{aligned}$$

provided $\theta \le \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2} - \delta\}$, $\delta > 0$, arbitrarily small. Summarizing the estimates above, we conclude that for $\gamma > \frac{3}{2}$ and

$$\begin{aligned}&\theta \le \frac{2}{3}\gamma - 1, \quad \gamma < 6\\ &\theta < \frac{\gamma}{2}, \quad \gamma \ge 6\end{aligned} \tag{4.3}$$

we have

$$\int_I \int_\Omega \eta \varrho_n^\gamma S_\alpha(b_k(\varrho_n))\psi \, \mathrm{d}x\mathrm{d}t \le C, \tag{4.4}$$

where the constant C is independent of α, k and n. Note that by a very technical bootstrap argument, we could improve the estimate for $\gamma \ge 6$ up to $\theta < \frac{2}{3}\gamma - 1$. Such an estimate however, is not needed to get the results presented in this paper, and we skip the details. We may thus pass with α to zero in (4.4) to get

$$\int_I \int_\Omega \eta \varrho_n^\gamma b_k(\varrho_n)\psi \, \mathrm{d}x\mathrm{d}t \le C$$

and finally, using the monotone convergence theorem, we may pass with $k \to \infty$. Thus

$$\int_I \int_\Omega \eta \varrho_n^{\gamma+\theta} \psi \, \mathrm{d}x\mathrm{d}t \le C.$$

As $\eta = 1$ in $[\frac{1}{2}, \frac{5}{2}]$, we get in particular

$$\|\varrho_n \psi\|_{L^{\gamma+\theta}((\frac{1}{2},\frac{5}{2})\times\Omega)} \le C(\psi), \tag{4.5}$$

provided $\gamma > \frac{3}{2}$.

With estimate (4.5) in hand, we can pass with n to infinity in the momentum equation. We get (for a suitably chosen subsequence, if necessary)

$$\varrho_{n_k} - \varrho \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L^r\left(\left(\frac{1}{2}, \frac{5}{2}\right); L_2^\gamma(\Omega)\right)$$
$$(\varrho_{n_k})^\gamma \rightharpoonup \overline{p} \quad \text{in } L^{\frac{\gamma+\theta}{\gamma}}\left(\left(\frac{1}{2}, \frac{5}{2}\right) \times \Omega'\right),$$

Ω' any bounded subdomain of Ω and $1 \le r < \infty$. The limit functions solve

$$\nabla \overline{p} = \varrho_s \nabla F$$
$$\varrho_s - \varrho_\infty \in L_2^\gamma(\Omega).$$

The next task is to show that $\overline{p} = \varrho_s^\gamma$, which is equivalent to the strong convergence of the density. It is again more or less a standard problem in the theory of compressible Navier–Stokes equations. Following the general argument of [9], we take $\alpha \in (0, \min\{\frac{1}{2} + \frac{\theta}{2\gamma}, \frac{\theta}{\gamma+\theta}\})$ and define

$$G(s) = s^\alpha, \quad s \in \mathbb{R}_0^+.$$

Then the function $G(p(\cdot))$ is a suitable function in the renormalized continuity equation. For the sake of simplicity, we again denote the subsequence ϱ_{n_k} constructed above by ϱ_n. Using the renormalized continuity equation (2.1) with $b(\varrho) = G(p(\varrho_n))$ and due to fact that $\nabla \boldsymbol{u}_n \to \boldsymbol{0}$ in $L^2((\frac{1}{2}, \frac{5}{2}) \times \Omega)^9$, we have that

$$\mathrm{Div}_{t,x}[G(p(\varrho_n)), G(p(\varrho_n))\boldsymbol{u}_n] = (G(p(\varrho_n)) - \varrho_n G'(p(\varrho_n)))\mathrm{div}\, \boldsymbol{u}_n$$

which is bounded in $L^p_{\mathrm{loc}}((\frac{1}{2}, \frac{5}{2}) \times \Omega)$, $\frac{1}{p} = \frac{1}{2} + \frac{\alpha\gamma}{\gamma+\theta}$, and therefore also precompact in $W^{-1,q_1}_{\mathrm{loc}}((\frac{1}{2}, \frac{5}{2}) \times \Omega)$ with $q_1 > 1$ suitably chosen. Here, the operator $\mathrm{Div}_{t,x}[V_0, V_1, V_2, V_3] = \frac{\partial V_0}{\partial t} + \sum_{i=1}^3 \frac{\partial V_i}{\partial x_i}$.

Next, we look at the momentum equation $(1.1)_2$. Thanks to the estimates of the density ϱ_n and the strong convergence of $\nabla \boldsymbol{u}_n$ mentioned above, we have that

$$\mathrm{Curl}_{t,x}[p(\varrho_n), 0, 0, 0] \quad \text{is precompact in } \left(W^{-1,q_2}_{\mathrm{loc}}\left(\left(\frac{1}{2}, \frac{5}{2}\right) \times \Omega\right)\right)^{16},$$

where $(\mathrm{Curl}_{t,x}[V_0, V_1, V_2, V_3])_{i,j} = \frac{\partial V_j}{\partial x_i} - \frac{\partial V_i}{\partial x_j}$, $x_0 = t$, $i, j = 0, \ldots, 3$. Indeed, the only non-zero term is $-\nabla p(\varrho_n) = \frac{\partial}{\partial t}(\varrho_n \boldsymbol{u}_n) + \mathrm{div}(\varrho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n) - \mu \Delta \boldsymbol{u}_n - (\mu + \lambda)\nabla \mathrm{div}\, \boldsymbol{u}_u + \varrho_n \boldsymbol{f}$.

Further, let us assume

$$p(\varrho_n) \rightharpoonup \overline{p} \quad \text{in } L^{r_1}_{\mathrm{loc}}\left(\left(\frac{1}{2},\frac{5}{2}\right)\times\Omega\right), r_1 = \frac{\gamma+\theta}{\gamma}$$
$$G(p(\varrho_n)) \rightharpoonup \overline{G(p)} \quad \text{in } L^{r_2}_{\mathrm{loc}}\left(\left(\frac{1}{2},\frac{5}{2}\right)\times\Omega\right), r_2 = \frac{1}{\alpha}$$
$$G(p(\varrho_n))p(\varrho_n) \rightharpoonup \overline{G(p)p} \quad \text{in } L^{q}_{\mathrm{loc}}\left(\left(\frac{1}{2},\frac{5}{2}\right)\times\Omega\right), \frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{q}<1.$$

Using the L^p-version of the div–curl lemma by Murat [10] and Tartar [11], we deduce

$$\overline{G(p)}\,\overline{p} = \overline{G(p)p}.$$

Hölder's inequality yields

$$\overline{p}^{1+\alpha} \le \overline{p^{1+\alpha}} = \overline{G(p)p} = \overline{G(p)}\,\overline{p} = \overline{p^\alpha}\,\overline{p} \le \overline{p}^\alpha\,\overline{p} = \overline{p}^{1+\alpha};$$

thus

$$\overline{p^{1+\alpha}} = \overline{p}^{1+\alpha}$$

and due to the uniform convexity of $L^{1+\alpha}((1,2)\times\Omega')$, $\overline{p} = \varrho_s^\gamma$, which means that

$$\varrho_n \to \varrho_s \quad \text{in } L^r_{\mathrm{loc}}((1,2)\times\Omega')$$

for any $1 \le r < \gamma+\theta$ and Ω' any bounded subset of Ω. Thus, we have the following lemma.

Lemma 3. *Under the assumptions of Theorem 2, from any sequence ϱ_n, we can choose a subsequence ϱ_{n_k} such that*

$$\varrho_{n_k} - \varrho_\infty \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L^s((1,2); L^\gamma_2(\Omega))$$
$$\varrho_{n_k} \to \varrho_s \quad \text{in } L^r((1,2)\times\Omega'),$$

$1 \le s < \infty$, $1 \le r < \gamma+\theta$, $\Omega' \subset \Omega$ any bounded subdomain. Moreover, $\varrho_s = \varrho_s(x)$ and it solves the problem

$$\begin{aligned} &\nabla(\varrho_s^\gamma) = \varrho_s \nabla F \quad \text{in } \mathcal{D}'(\Omega)\\ &\varrho_s - \varrho_\infty \in L^\gamma_2(\Omega)\\ &\varrho_s \ge 0 \quad \text{a.e. in } \Omega. \end{aligned} \tag{4.6}$$

5. Stationary problem

In this section, we would like to find a sufficiently large class of potentials F for which problem (4.6) is uniquely solvable in a certain reasonable regularity class. Problem (4.6) with $\varrho_\infty = 0$ if Ω is an exterior domain, or for the case Ω bounded, can be found in [3] or [12]. To the knowledge of the authors, the problem Ω exterior and $\varrho_\infty > 0$ has not been studied yet.

Let us recall that we assume conditions (2.8) and (2.9). For a moment, let also $F > -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}$ in Ω. Then, following the above mentioned papers, it is easy to verify that in the class $RC = \{v \in L^\infty(\Omega); v \ge 0 \text{ a.e. in } \Omega\}$, the only solution to (4.6) is

$$\varrho_s = \left[\frac{\gamma-1}{\gamma}F + \varrho_\infty^{\gamma-1}\right]^{\frac{1}{\gamma-1}}. \tag{5.1}$$

Evidently, under the assumptions on F (see (2.9)), we have $\varrho_s \in L^\infty(\Omega)$; further $\varrho_s - \varrho_\infty \sim F$ as $|x| \to \infty$ (recall that $F(x) \to 0$ as $|x| \to \infty$), and thus $\varrho_s - \varrho_\infty \in L^2(\Omega) \cap L^\infty(\Omega)$. Furthermore, as the set $\{x \in \Omega; |\varrho_s - \varrho_\infty| > 1\}$ is bounded, $\varrho_s - \varrho_\infty$ belongs in particular to $L^\gamma_2(\Omega)$ for any $\gamma \in [1,\infty)$. The uniqueness is evident (recall that other candidates for being the solution, functions of the type $([\frac{\gamma-1}{\gamma}F + C]^+)^{\frac{1}{\gamma-1}}$, do not satisfy $\varrho_s - \varrho_\infty \in L^\gamma_2(\Omega)$, except for $C = \varrho_\infty^{\gamma-1}$).

We have constructed one class of potentials for which (4.6) has a unique solution. Note that for its uniqueness, we do not need any further information on ϱ_s (like the value of the energy or the mass, as was the case in [3] or [12]). The sole condition that $\varrho_s - \varrho_\infty \in L^2(\Omega)$ (which is actually equivalent to $\varrho_s - \varrho_\infty \in L_2^\gamma(\Omega)$) is sufficient.

We would also like to consider certain potentials that may assume values below $-\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}$. It is not very surprising that generally, the uniqueness of the solution may be lost. To see this, let us consider for simplicity the one-dimensional case with the potential as shown in Fig. 1.

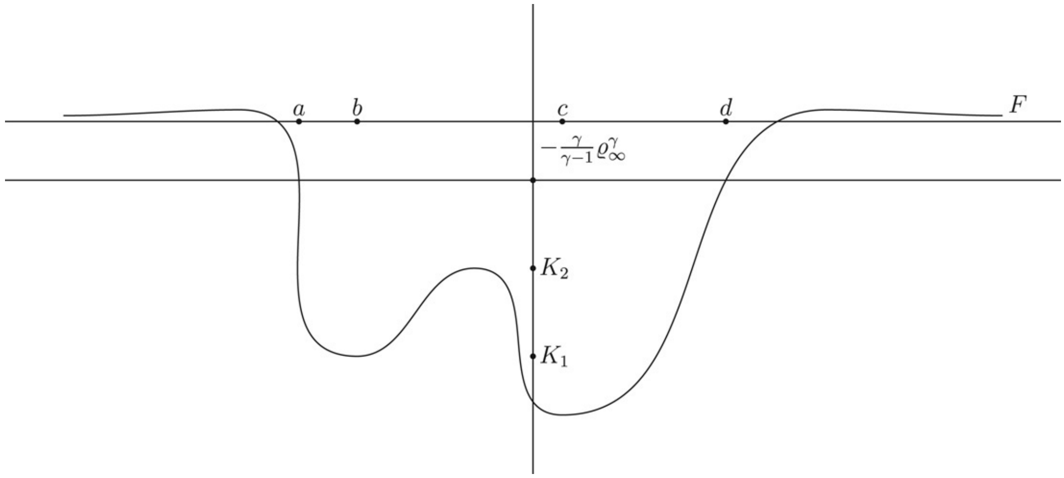

Fig. 1.

In this case, we have a continuum of solutions. Any function of the type

$$\varrho_s = \begin{cases} \left(\dfrac{\gamma-1}{\gamma}F + \varrho_\infty^{\gamma-1}\right)^{\frac{1}{\gamma-1}} & x \in \mathbb{R} \setminus (a, d) \\ 0 & x \in [a, b] \cup [c, d] \\ \left(\left[\dfrac{\gamma-1}{\gamma}(F+K) + \varrho_\infty^{\gamma-1}\right]^+\right)^{\frac{1}{\gamma-1}} & x \in (b, c),\ K \in \left[0, -\dfrac{\gamma-1}{\gamma}\varrho_\infty^{\gamma-1} - K_1\right] \end{cases}$$

is a solution to (4.6).

This example of non-uniqueness, similar to those in [3] or [12], is based on the fact that the level sets $\{x \in \Omega;\ F(x) > k\}$ are not connected for some values of k. A similar situation as in Fig. 1 may occur only in a bounded part of Ω, because $F \to 0$ as $|x| \to \infty$.

If we exclude such situations, we may still show the uniqueness of the solution even though $F(x)$ may be less than $-\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}$ in a certain (necessarily bounded!) part of Ω.

Lemma 4. *Suppose that F satisfies (2.8) and (2.9). Then, in the class $RC = \{v \in L^\infty(\Omega);\ v \geq 0$ a.e. in $\Omega\}$ there exists just one solution to (4.6), and it is given by*

$$\varrho_s = \left(\left[\frac{\gamma-1}{\gamma}F + \varrho_\infty^{\gamma-1}\right]^+\right)^{\frac{1}{\gamma-1}}. \tag{5.2}$$

Proof. Evidently, any solution to (4.6) from RC must be continuous in $\overline{\Omega}$. Repeating now step by step arguments from [12], we conclude that in any connected region where $\varrho_s > 0$, the solution is given by

$$\varrho_s = \left(\frac{\gamma-1}{\gamma}F + K\right)^{\frac{1}{\gamma-1}}, \tag{5.3}$$

where K is a suitable constant. Since $\varrho_s - \varrho_\infty \in L_2^\gamma(\Omega)$, $F \to 0$ as $x \to \infty$ and $\Omega_\infty = \{x \in \Omega; F(x) > -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}\}$ is connected, it follows that

$$\varrho_s = \left(\frac{\gamma-1}{\gamma}F + \varrho_\infty^{\gamma-1}\right)^{\frac{1}{\gamma-1}}$$

in Ω_∞. It remains to verify that under assumption (2.9), $\varrho_s = 0$ in the remaining part of Ω.

First, from the continuity of ϱ_s, it follows that $\varrho_s = 0$ at $\partial\Omega_\infty$. Assume that there is $\tilde{x} \in \Omega\backslash\Omega_\infty$ such that $\varrho_s(\tilde{x}) > 0$. Take a maximally connected component Ω_1 of $\Omega \setminus \Omega_\infty$ such that $\tilde{x} \in \Omega_1$ and $\varrho_s > 0$ in Ω_1. Then ϱ_s is given in Ω_1 by formula (5.3) with $K > \varrho_\infty^{\gamma-1}$, and due to the continuity of ϱ_s and F, $\mathrm{dist}(\Omega_1, \Omega_\infty) > 0$. We also have $F > -\frac{\gamma}{\gamma-1}K$ in Ω_1 and $F = -\frac{\gamma}{\gamma-1}K$ at $\partial\Omega_1$, since $\varrho_s = 0$ here. But this contradicts (2.9) since $F = -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1} > -\frac{\gamma}{\gamma-1}K$ at the boundary of any connected component $\tilde{\Omega}_i$ of $\{x \in \Omega; F(x) < -\frac{\gamma}{\gamma-1}\varrho_\infty^{\gamma-1}\}$. □

6. Proof of Theorem 2

Take any sequence $\tau_n \to \infty$. We know that we can extract a subsequence τ_{n_k} such that

$$\begin{aligned} &\nabla \boldsymbol{u}_{n_k} \to 0 \quad \text{in } (L^2((1,2)\times\Omega))^9 \\ &\varrho_{n_k} - \varrho_\infty \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L^s((1,2); L_2^\gamma(\Omega)),\ 1 \le s < \infty \\ &\varrho_{n_k} \to \varrho_s \quad \text{in } L^r((1,2)\times\Omega'),\ 1 \le r < \gamma+\theta, \end{aligned} \tag{6.1}$$

where Ω' denotes here, and in what follows, any bounded subdomain of Ω. Thanks to the uniqueness of the limit function ϱ_s (any limit function must be a solution to (4.6)), we infer that the whole sequence is convergent in the sense of (6.1). To finish the proof of Theorem 2, we have to show that (6.1) implies pointwise convergence ($t \to \infty$)

$$\begin{aligned} &\varrho(t,\cdot) - \varrho_\infty \rightharpoonup \varrho_s - \varrho_\infty \quad \text{in } L_2^\gamma(\Omega) \\ &\varrho(t,\cdot) \to \varrho_s \quad \text{in } L^r(\Omega'),\ 1 \le r < \gamma \\ &(\varrho\boldsymbol{u})(t,\cdot) \rightharpoonup \boldsymbol{0} \quad \text{in } (L_2^{\frac{2\gamma}{\gamma+1}}(\Omega))^3. \end{aligned} \tag{6.2}$$

Let us start with the weak convergence of the density. Take any $\varrho \in C_0^\infty(\Omega)$. Thanks to $(1.1)_2$ and (3.5) $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\varrho_n - \varrho_\infty)\varphi\mathrm{d}x$ and $\int_\Omega(\varrho_n - \varrho_\infty)\varphi\mathrm{d}x$ are bounded in $L^\infty(1,2)$. Therefore by the Sobolev imbedding theorem and by the Arzelà–Ascoli theorem we deduce

$$\int_\Omega(\varrho(t,\cdot) - \varrho_\infty)\varphi\mathrm{d}x \to \int_\Omega(\varrho_s(\cdot) - \varrho_\infty)\varphi\mathrm{d}x \quad \text{in } C^0[1,2]. \tag{6.3}$$

Finally, we observe by the density argument that (6.3) holds for any $\varphi \in L_2^{\gamma'}(\Omega)$, and thus $(6.2)_1$ holds true.

The proof of $(6.2)_3$ follows the same idea; only instead of the continuity equation, we use the momentum equation $(1.1)_1$ and the boundedness of $\varrho_n\boldsymbol{u}_n$ in $L^\infty((1,2); (L_2^{\frac{2\gamma}{\gamma+1}}(\Omega))^3)$.

Finally, let us prove $(6.2)_2$. We show by Lemma 3 that

$$\sqrt{\varrho_n} \to \sqrt{\varrho_s} \quad \text{in } L^r((1,2)\times\Omega'),\ 1 \le r < 2(\gamma+\theta).$$

As $\sqrt{\varrho}$ satisfies the renormalized continuity equation (2.1), we show as above that for $t \to \infty$,

$$\sqrt{\varrho(t,\cdot)} \rightharpoonup \sqrt{\varrho_s} \quad \text{in } L^{2\gamma}(\Omega').$$

Weak convergence $(6.2)_1$ implies

$$\|\sqrt{\varrho(t,\cdot)}\|_{L^2(\Omega')} \to \|\sqrt{\varrho_s}\|_{L^2(\Omega')} \quad \text{as } t \to \infty.$$

Since L^2 is uniformly convex, the last two formulae yield

$$\sqrt{\varrho(t,\cdot)} \to \sqrt{\varrho_s} \quad \text{in } L^2(\Omega')$$

as $t \to \infty$. As ϱ is bounded in $L^\infty((0,\infty); L^\gamma(\Omega'))$, $(6.2)_2$ follows easily and thus Theorem 2 is proved.

We shall finish this paper by remarking that condition (1.5) can be weakened. In fact, we can replace it by $p \in C[0,\infty) \cap C^1(0,\infty)$, $p(0) = 0$, $p' > 0$, $\int_0^1 \frac{p'(s)}{s} \mathrm{d}s < \infty$ and $p(s) \sim s^\gamma$ for s large.

In this case, the function $P(\varrho)$ has to be replaced by

$$P_1(\varrho) = \varrho \int_{\varrho_\infty}^{\varrho} \frac{p(s)}{s^2} \mathrm{d}s - \frac{\varrho}{\varrho_\infty} p(\varrho_\infty) + p(\varrho_\infty).$$

The properties of $P_1(\varrho)$ are the same as those of $P(\varrho)$; in particular, $P_1(\varrho) \sim (\varrho - \varrho_\infty)^2$ as $|\varrho - \varrho_\infty| \le 1$, and $P_1(\varrho) \sim |\varrho - \varrho_\infty|^\gamma$ as $|\varrho - \varrho_\infty| > 1$. Instead of (4.6), we have to solve

$$\begin{aligned} &\nabla(p(\varrho_s)) = \varrho_s \nabla F \quad \text{in } \Omega \\ &\varrho_s - \varrho_\infty \in L_2^\gamma(\Omega) \\ &\varrho_s \ge 0 \quad \text{a.e. in } \Omega \end{aligned} \tag{6.4}$$

and under the assumptions on F from Lemma 4, modified with respect to the function $p(\varrho)$, the unique solution to (6.4) is of the form

$$\varrho = G^{-1}([F + G(\varrho_\infty)]^+),$$

where $G(r) := \int_0^r \frac{p'(s)}{s} \mathrm{d}s$. The strict monotonicity of p ensures the existence of G^{-1}. The details are left to the reader as an exercise.

Acknowledgements

The second author was supported by the Grant Agency of the Czech Republic (grant No. 201/03/0934 and grant No. 201/02/P091) and by the Council of the Czech Government (project No. MSM 0021620839).

References

[1] A. Novotný, I. Straškraba, Stabilization of weak solutions to compressible Navier–Stokes equations, J. Math. Kyoto Univ. 40 (2) (2000) 217–245.
[2] A. Novotný, I. Straškraba, Convergence to equilibria for compressible Navier–Stokes equations with large data, Ann. Mat. Pura Appl. 179 (4) (2001) 263–287.
[3] E. Feireisl, H. Petzeltová, Large time behaviour of solutions to the Navier–Stokes equations of compressible flow, Arch. Ration. Mech. Anal. 150 (1) (1999) 77–96.
[4] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flows, Oxford University Press, Oxford, 2004.
[5] P.L. Lions, Mathematical Topics in Fluid Dynamics. Vol. 2. Compressible Models, Oxford University Press, New York, 1998.
[6] S. Fučík, O. John, A. Kufner, Function Spaces, Academia, Praha, 1977.
[7] E. Feireisl, On compactness of solutions to the compressible isentropic Navier–Stokes equations when the density is not square integrable, Comment. Math. Univ. Carolin. 42 (1) (2001) 83–98.
[8] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004.
[9] E. Feireisl, H. Petzeltová, On the zero-velocity-limit solutions to the Navier–Stokes equations of compressible flow, Manuscripta Math. 97 (1) (1998) 109–116.
[10] F. Murat, Compacité par compensation, Ann. Sc. Norm. Super. Pisa 5 (4) (1978) 489–507.
[11] L. Tartar, Compensated compactness and applications to partial differential equations, in: R.J. Knopps (Ed.), Nonlinear Anal. and Mech., Pitman, London, 1979, pp. 136–212.
[12] R. Erban, On the static-limit solutions to the Navier–Stokes equations of compressible flow, J. Math. Fluid Mech. 3 (4) (2001) 393–408.